ISSN: 2148-9955

International Journal of Research in Education and Science (IJRES)

www.ijres.net

Prospective Middle-School Mathematics Teachers' Quantitative Reasoning and Their Support for Students' Quantitative Reasoning

Tangul Kabael [1], Ayca Akin[2]
[1]Anadolu University
[2]Kastamonu University

To cite this article:

Kabael, T. & Akin, A. (2018). Prospective middle-school mathematics teachers' quantitative reasoning and their support for students' quantitative reasoning. *International Journal of Research in Education and Science (IJRES), 4*(1), 178-197. DOI:10.21890/ijres.383126

IJRES International Journal of Research in Education and Science
Volume 4, Issue 1, Winter 2018 DOI:10.21890/ijres.383126

Prospective Middle-School Mathematics Teachers' Quantitative Reasoning and Their Support for Students' Quantitative Reasoning

Tangul Kabael, Ayca Akin

Article Info

Article History

Received:
21 March 2017

Accepted:
02 July 2017

Keywords

Quantitative reasoning
Problem solving
Prospective middle-school mathematics teachers
Questioning
Guidance

Abstract

The aim of this research is to examine prospective mathematics teachers' quantitative reasoning, their support for students' quantitative reasoning and the relationship between them, if any. The teaching experiment was used as the research method in this qualitatively designed study. The data of the study were collected through a series of exploratory teaching interviews and debriefing interviews with nine focus group participants, and clinical interviews that the participants conducted with middle-school students. The results indicated that the participants with strong quantitative reasoning use problem-solving approaches that focused on the quantity, whereas the participants with poor quantitative reasoning use problem-solving approaches that focused on performing calculations, using formulas and procedures devoid of quantitative meaning in solving of the problem. During the questioning process, the participants with strong quantitative reasoning led their students to identify and interpret the quantities, determine relationships among the quantities, represent all the quantities and their interrelationships, whereas the participants with poor quantitative reasoning led their students to perform calculations, make algebraic manipulations and focus on numbers by ignoring the quantities in the problem. These results suggest that prospective mathematics teachers' quantitative reasoning is strongly associated with their support for students' quantitative reasoning in the problem-solving process.

Introduction

Problem solving has been studied in the field of mathematics education for many years and there are many studies on problem solving in the literature. A first study of the problem-solving framework was developed by Polya (1957). After Polya's studies on problem solving, this subject has become a field of study on which mathematics educators overemphasize. Many studies in the literature characterize the mental actions of students with regard to the problem-solving phases (i.e. understanding the problem, devising a plan, carrying out the plan and looking back) identified by Polya (1957). Several other frameworks (e.g. Carlson & Bloom, 2005; Schoenfeld, 2007) have contributed to the problem-solving literature after the problem-solving framework developed by Polya. Problem-solving has significant place in mathematics teaching and learning. Problem-solving is not only a purpose in school mathematics but also the meaning of mathematics learning (Harel, 2007). Although getting correct answer for the problem is praised as success in school mathematics, problem-solving strategies and the ways of reasoning in the problem-solving process should be focused on (Harel, 2007). Considering problem solving in terms of the development of reasoning, problem solving has a notable value in middle-school mathematics. Middle-school mathematics covers the transition from arithmetic to algebra, and it is necessary to provide students with many problem-solving experiences to promote their algebraic reasoning in this process (Cai & Knuth 2011). Moreover, quantitative reasoning plays a key role in the improvement of algebraic reasoning in the transition from arithmetic to algebra. Problem solving can be used as a meaningful tool for developing students' mathematical reasoning such as quantitative reasoning (Cai & Nie, 2007). Smith and Thompson (2008) state that when a middle-school student reasons quantitatively in the problem-solving process, this allows her/his to better understand the problem and to take more advantage of her/his everyday experiences. Akkan, Baki and Çakıroğlu stress that problem solving plays an important role in overcoming middle-school students' difficulties arising in the transition from arithmetic to algebra and enables them to reason quantitatively. In conclusion, reasoning quantitatively enables middle school students both to support a smooth transition from arithmetic to algebra and to solve algebraic verbal problems productively (Ellis, 2007; Smith & Thompson, 2008).

There are many definitions in the literature for quantitative reasoning. According to Weber, Ellis, Kulow and Ozgur (2014), quantitative reasoning is a way of reasoning related with identifying the quantities in the problem, determining and coordinating the quantities and interrelationships among them, considering the unit of quantities, and executing calculations with referencing the quantities in the plan or model of the problem. Moore, Carlson and Oehtman (2009) remark that quantitative reasoning includes the mental actions like understanding the context of the problem, constructing the quantities of the conceived situation, determining the relationships among the quantities and analyzing the change in quantities. In short, quantitative reasoning is to reason with quantities and their interrelationships (Ellis, 2009). Kaput (1995) defines quantitative reasoning as a network of quantities and quantitative relationships in the context of any situation. The students construct the quantities in the context of the problem when they reason quantitatively. Quantities are properties of an object or phenomenon, which can be measured (Weber et al., 2014). Therefore, the act of measuring makes properties of an object or phenomenon, which can be measured, quantities (Smith & Thompson, 2008). In the construction process of a quantity, an individual should be aware of the object, the measurable attribute of the object, the appropriate unit of the measurable attribute and assigning a numerical value to measurable quality or measurable characteristic of an object (Kaput, 1995). For this reason, when a student reasons quantitatively, s/he is able to use a number of skills such as constructing the quantity, determining relationships among the quantities, coordinating the dependence of one quantity on another quantity, and making quantitative deductions. Since quantitative reasoning is a skill that needs to be improved from the pre-school, many components of quantitative reasoning are used in the construction process of the concept of quantity. Furthermore, students engage in many components of quantitative reasoning such as constructing the quantity in the context of the problem, determining and representing relationships among the quantities, analyzing the change in quantities, making quantitative deductions, making a quantitative-unit coordination and constructing a quantitative-unit conservation in the middle-school mathematics.

Problem solving, and quantitative reasoning play a key role in the improvement of algebraic reasoning in middle-school mathematics. Quantitative reasoning plays the role as glue during this process, providing a smooth and soft transition from arithmetic reasoning to algebraic reasoning (Smith & Thompson, 2008). Ellis (2007) points out that quantitative reasoning provides a basis for algebraic reasoning by serving as a bridge between arithmetic and algebra. Promoting the middle school students' quantitative reasoning may assist in the development of their mathematics concepts and skills (Smith & Thompson, 2008). Therefore, it is necessary to give importance to the development of quantitative reasoning in order that middle-school students can learn mathematics meaningfully. Furthermore, the curricula of middle-school mathematics and several researches indicates that quantitatively-rich problems enable students to promote algebraic reasoning (Ellis, 2007). On the other hand, Ellis (2007) stresses that solving quantitatively-rich problems does not guarantee improvement in students' quantitative reasoning and problem solving skills, and does not function as a cure-all for problems arising in the problem-solving process. For this reason, middle-school mathematics teachers should lead students to construct quantities, determine relationships among the quantities, coordinate the dependence of one quantity on another quantity, explore emergent quantities and quantitative meanings in the problem-solving process in order to develop the quantitative reasoning of them (Ellis, 2007; Weber et al., 2014). In the problem-solving process, the use of well-structured guidance or questioning in terms of quantitative meaning enables middle-school students to both reason quantitatively and facilitate their transition from arithmetic to algebra (Ellis, 2007; Smith & Thompson, 2008). Moreover, questioning or guidance in terms of quantitative reasoning is used by qualified mathematics teachers. For example, a mathematics teacher, who aims to improve middle-school students' quantitative reasoning in the problem-solving process, should focus on and give importance constructing the concept of quantity and identifying the quantities in the problem situation. Since the quantities are cognitive structures, their construction requires a great effort (Thompson, 2011).

As mentioned in the literature, these highlights indicate that the teacher plays an important role in the development of students' quantitative reasoning in the problem-solving process. Middle-school mathematics teachers have many responsibilities in the problem-solving process. One of the most critical responsibilities of middle-school mathematics teachers is unquestionably support students' reasoning about the problem situation (Polya, 1957). Rigelman (2007) argues that teachers have to productively conduct the problem-solving activity by asking questions to elicit students' reasoning. Ellis (2007) remarks that promoting the middle-school students' quantitative reasoning in learning environment depends highly on middle-school mathematics teachers who are able to use well-structured questioning in terms of quantitative meanings and structures in the problems. On the other hand, Moore (2011) emphasizes that a leading, which is deficient in respect of quantitative meanings will not assist in both solving algebraic-verbal problems and supporting the development of reasoning and knowledge structures of students. Weber et al. (2014) recommend teachers to use practical tips to help students develop quantitative reasoning, since quantitative reasoning has a critical importance in the middle-school mathematics. Recent studies stress (e.g. Moore, 2011; Weber et al., 2014) that teachers should lead middle-school students by focusing on the quantities in order to develop students' quantitative reasoning in the problem-solving process.

On the other hand, the mathematical content knowledge of the mathematics teacher is a crucial prerequisite in constructing pedagogical content knowledge (van den Kieboom, Magiera, & Moyer, 2014). Baumert et al. (2010) emphasize that sufficient content knowledge of mathematics is essential for enhancing pedagogical content knowledge. For this reason, insufficient mathematical content knowledge is correlated with the lack of pedagogical content knowledge (Van den Kieboom et al., 2014). In other respects, several studies (e.g. Watson & Harel, 2013) indicate that mathematics teachers with sufficient mathematical knowledge have significant effects on their teaching at middle school level. The results of Goe's (2007) study reveal that the quality of the teachers' content knowledge and pedagogical knowledge of mathematics is positively associated with students' mathematics performance in all grades but especially at the middle-school level. One of the components of middle school teachers' mathematical content knowledge is their quantitative reasoning, whereas one of the components of their pedagogical mathematical knowledge is their support for middle-school students' quantitative reasoning. Therefore, prospective middle-school mathematics teachers' quantitative reasoning and their support for students' quantitative reasoning become important in the problem-solving process. When considering the role of middle school mathematics teachers' in developing students' problem-solving skills and quantitative reasoning, the recent studies on problem solving have revealed the importance of mathematics teachers' quantitative reasoning, and their supportive guidance and questioning for middle-school students' quantitative reasoning. Despite there are a few studies regarding teachers' and prospective teachers' questioning ability in the literature (e.g. Moyer & Milewicz, 2002), there is scarcely any study on the relationship between prospective teachers' mathematical content knowledge and questioning ability (van den Kieboom et al., 2014). For example, van den Kieboom et al. (2014) explore the relationship between prospective teachers' algebraic thinking proficiency and questioning ability. Furthermore, no study has been encountered regarding prospective middle-school mathematics teachers' mathematical-content knowledge in terms of quantitative reasoning and their support for middle-school students' quantitative reasoning. This study is focused on prospective middle-school mathematics teachers' quantitative reasoning in the problem-solving process, their conceptions regarding quantity, quantitative reasoning and the development of quantitative reasoning and their support for middle-school students' quantitative reasoning in the problem-solving process based on Polya's problem-solving framework and on the literature on quantitative reasoning. Due to the reasons mentioned above, it is thought that the study may enrich the relevant literature.

Aim of the Study

The aim of this research is to investigate prospective middle-school mathematics teachers quantitative reasoning in the problem-solving process, how they support for middle school students' quantitative reasoning and the relationship between their quantitative reasoning and ways of supporting middle school students' quantitative reasoning in the problem-solving process, if any.

Research Questions

(i) How proficiently do prospective middle-school mathematics teachers reason quantitatively in the problem-solving process?
(ii) How prospective middle-school mathematics teachers' ways of supporting middle-school students' quantitative reasoning are in the problem-solving process?
(iii) If any, what is the relationship between prospective middle school mathematics teachers' quantitative reasoning and their ways of supporting middle school students' quantitative reasoning in the problem-solving process?

Method

This study was carried out within the scope of the elective course offered in middle school mathematics teaching program in faculty of education at a state university. This study designed as a teaching experiment, as described by Steffe and Thompson (2000). The teaching experiment methodology, which is used especially in the field of mathematics and science education, provides the researcher the opportunity to describe and model participants' emerging reasoning and behaviors and to gain experience of this process (Steffe & Thompson, 2000).

Participants

The study was conducted in a middle-school mathematics teacher education program at a state university in Turkey. 28 prospective middle-school mathematics teachers enrolled in the teaching-experiment and nine out of these 28 prospective teachers were selected by using purposive sampling method (Fraenkel & Wallen, 1996) in this study. By using purposive sampling method, the participants were selected based on criteria which were their good class attendance, their ability to verbalize their thought processes and voluntariness. The participants of the study who were at the junior or senior undergraduate level had previously taken at least five mathematics content courses and took or have taken two semesters of special teaching methods course.

The Process of the Teaching Experiment

The elective course in which the study was conducted is instructed by the first author as an elective course in the middle school mathematics teaching programme of a state university. The teaching experiment consisted of twelve 120-minute teaching sessions within a span of fourteen weeks. Each teaching-experiment session included all 28 students, the instructor (first author), and an observer (second author). Promoting the prospective teachers' quantitative reasoning and developing their pedagogical approaches in relation with the supporting middle-school students' quantitative reasoning are among the objectives of this course. In the first three weeks of the course, theoretical knowledge associated with algebraic reasoning, problem solving and quantitative reasoning in middle school mathematics was given to the prospective teachers. In the next three weeks, instructional strategies were presented to prospective teachers to support middle-school students' these skills in the problem-solving process and the solution process of various algebraic-verbal problems were discussed with prospective teachers. At the end of six weeks, a training on the clinical-interview process and analyzing transcript of clinical interview for the participants was provided. Then these prospective teachers were asked to choose a middle-school grade and they were given two problems (see, appendix) that were prepared for their chosen middle-school grades. The prospective teachers were requested to conduct clinical interviews with a middle school student in the scope of provided these algebraic-verbal problems. Furthermore, the participants were asked to get permission from the parents of the middle school students they interviewed with, and get a parent permission form signed by a parent. The problems used in clinical interviews that are similarly used in the literature represent quantitatively-rich problems. Each interview conducted by prospective teachers lasted for between 45 and 65 minutes. Each prospective teacher, who completed the clinical interview and its analysis within six weeks, presented the results of her/his analysis in class. Moreover, the prospective teachers were required to submit their work (video recording, transcript of clinical interview and its analysis, and the work sheets used by the students in the problem-solving process) to the researchers.

Data Collection

Considering nature of the teaching-experiment method, data of the study were collected through a series of exploratory teaching interviews, debriefing interviews with focus group participants, and clinical interviews that the participants conducted with middle school students. The clinical-interview technique developed by Piaget aimed to investigate individuals' mental activities and ways of reasoning in-depth. The clinical-interview technique gave mathematics education researchers an opportunity to observe students' behaviors in the problem-solving process and make an inference about the changes in students' cognitive and affective structures (Goldin, 2000). For the clinical-interview technique, the researcher designs an open-ended problem and possible main questions associated with this problem in advance. However, the order of questions and sub-questions prepared for the clinical-interview process is structured in respect to the participant's mental processes, considering think-aloud protocols (Clements, 2000; Koichu & Harel, 2007). The basic approaches and principles of the exploratory-teaching interview technique used in this study are the same as those of the clinical-interview technique. Individual interview conducted to elicit the development of each participant's mental process after teaching sessions is called as exploratory teaching interview (Moore, 2010). According to Moore (2011), exploratory teaching interviews help researchers to describe the participants' reasoning and to have an idea of their mental actions in the problem -solving process. Moreover, Moore (2010) emphasized that this interview technique gives an opportunity for the researcher to determine possible limitations to the participants' current ways of reasoning that may not be elicit in other interviews or classroom environment. The debriefing interview is used to reveal the participants' conceptions, beliefs or reflections associated with any teachings, subjects or concepts (McAlpine, Weston, & Beauchamp, 2002). The technique of this interview is in accordance with the basic approaches and principles of the semi-structured interview technique. In the problem-solving process, clinical-interview technique was preferred in order to be included in the interviewee's mental process in which the interviews conducted by the researchers with prospective teachers and by prospective teachers with one middle school student. Furthermore, the semi-structured interview technique was preferred to investigate prospective teachers' knowledge and awareness related to quantitative reasoning and supporting this reasoning in this study.

In the context of this research, three quantitatively-rich problems, which are frequently used to investigate prospective or undergraduate students' quantitative reasoning in the literature, were adapted and used in the exploratory-teaching interviews (see, appendix). In order to solve the box problem which was adapted from the study conducted by Moore and Carlson, students have to identify the quantities, imagine the interrelationships among the quantities, use the conceptual knowledge and create a formula in terms of quantitative meaning. A solution to the candle-burning problem, which was adapted from the study of Carlson (2013), is required to build mental models of the specific quantities, relationships among these quantities and to construct a formula that represents these relationships in terms of quantitative meaning. In an attempt to solve the coin problem, which was adapted from the study conducted Olive and Çağlayan (2008), students need to determine the quantities in the context of the problem and make a quantitative-unit coordination and construct a quantitative-unit conservation to determine relationships among the quantities. In the solution process of three quantitatively-rich problems, the prospective teachers are expected to use various components of quantitative-reasoning skills such as identifying the quantities, determining relationships among the quantities, analyzing the change in quantities, creating a formula in terms of quantitative meaning, making a quantitative-unit coordination and constructing a quantitative-unit conservation. For this reason, the prospective teachers need to use problem solving approaches that focused on the quantity in order to solve these problems. The focus group participants allocated approximately 45 minutes to solve each problem (total 135 minutes). At the end of teaching-experiment, all focus group participants attended a 30-min debriefing interview including questions about examining how their conceptions of quantity, quantitative reasoning and pedagogical approaches to support for middle school students' quantitative reasoning in the problem-solving process.

Data Analysis

The data were analyzed using an open and axial coding approach (Strauss and Corbin, 1998) and conceptual analysis (Thompson, 2000) that involved examining the relationship between the prospective middle-school mathematics teachers' quantitative reasoning and their support for students' quantitative reasoning in the problem-solving process. Since this study associated with quantitative reasoning in the problem-solving process, the studies investigating quantitative reasoning in the problem-solving were examined in the literature and two analysis frameworks were constructed.

While both analysis frameworks were constructing, Moore's (2011) study associated with problem solving activities and orientations in terms of the students' ability to engage in quantitative reasoning and Weber's et al. (2014) study related to practical tips suggested for teachers to help students develop quantitative reasoning were considered in this study. Because of the fact that Moore's (2011) study describes students' problem-solving behaviors in terms of quantitative reasoning during the problem-solving phases, the first analysis framework was constructed mainly based on Moore's study. Furthermore, for the first analysis framework, the researchers addressed Weber's et al. (2014) study's suggestions that how students should reason quantitatively in the modeling problem. For this purpose, the researchers primarily characterized the participants' mental actions in terms of quantitative reasoning in the problem-solving process. This phase of the data analysis involved the participants' disposition to engage in quantitative reasoning and their actions in the problem-solving process (see, adapted from Moore, 2011, pp. 309-310). The first analysis framework used to analyze prospective middle-school mathematics teachers' quantitative reasoning in the problem-solving process as follows.

Weber et al. (2014) recommend teachers to use practical tips to develop students' quantitative reasoning. These practical tips give information to the teachers about guiding students in terms of quantitative reasoning skill in the problem-solving process. These suggestions of Weber et al. (2014) constituted the focus of the second analysis framework in this study. Moreover, Moore's (2011) study which describes students' problem-solving behaviors in terms of quantitative reasoning was considered for the second analysis framework. The questioning skills of the participants were characterized in terms of practical tips for teachers to help students develop quantitative reasoning in the problem-solving process (see, adapted from Weber et al., 2014, pp.26-30). The second analysis framework used to analyze questioning skills of prospective middle-school mathematics teachers as follows.

Table 1. Features of the quantitative reasoning in problem solving

Characteristics of quantitative reasoning	Examples of problem-solving actions
Conceiving of a problem situation	Spending a significant amount of time describing the context of the problem, Spending a significant amount of time developing an image of the problem's context (e.g. drawing a figure (diagram) of the situation).
Identifying and interpreting quantities in the problem situation	Identify and interpret the quantities that problem solvers believe are related to solving the problem, Returning to the figure (diagram) of the situation to label determined values during the problem-solving process, Identify and interpret particular quantities and how they intend to or imagine measuring those quantities.
Making a plan in terms of quantitative relationships	Recalling a formula and describing it in terms of quantitative relationships, Determining meaningful relationships among the quantities in the situation, Representing all the quantities and their interrelationships in the model or formula, Determining how varying individual quantities affects the rest of the quantities in the model (i.e. quantitative coordination).
Executing calculations with respect to the quantities in the model or plan	Manipulating, and using the quantities to make a problem situation coherent, Describing calculations in terms of the quantities of the situation and relationships between these quantities, Executing and describing calculations with reference to the context of the problem.
Checking of calculations, as calculations were planned in terms of quantitative relationships or quantitative coordination	Revising and retesting aspects of their solution plan in terms of quantitative relationships, Testing relationships and then justifying why the relationships always or do not always hold, Exhibiting confidence in their solutions referencing to the context of the situation in term of quantities.

To characterize the participants' conceptions of quantity, quantitative reasoning and pedagogical approach to support for middle school students' quantitative reasoning in the problem-solving process, the definitions of these quantitative structures and related pedagogy identified by Kaput (1995), Smith and Thompson (2008) and Weber et al. (2014) were also used. The definition of quantity identified by Kaput (1995), Smith and Thompson (2008) and Weber et al. (2014) was handled. The definitions of quantitative reasoning and quantitative reasoning components identified by Weber et al. (2014) and Kaput (1995) were considered. Pedagogical approaches associated with quantitative reasoning suggested by Weber et al. (2014) and Kaput (1995) was handled. Studying of these characterizations provide opportunity of understanding how the reasoning and leading patterns associated with the participants' problem-solving actions, questioning and views are related to the researchers.

On the other hand, conceptual analysis is used in an attempt to portray and model the participants' actions (verbal and written products) and views. Conceptual analysis is a qualitative analytical technique which is used to portray and model each participant's changing and development of actions, ways of reasoning and conceptions from the beginning to the end of the teaching-experiment process (Thompson, 2000). In order to provide consistency and trustworthiness of the data analysis, each researcher analyze the data individually. Using the formula for credibility and trustworthiness as identified by Miles and Huberman (1994), the researchers obtained p = .94.

Table 2. Practical tips for teachers to help students develop quantitative reasoning

Characteristics of questioning, focusing on quantitative reasoning	Examples of problem-solving actions
Leading the students to identify and interpret the quantities in order that they must understand the problem.	Giving an opportunity to the students describes the context of the problem and constructing an image of the problem's context (e.g. drawing a diagram of the situation), Promoting the students to identify the quantities that are associated with the problem situation rather than prescribing the quantities for them, Asking questions about a problem that focus on how students intend to or imagine measuring the quantities and why they chose to identify particular quantities.
Leading the students to make a plan in term of determining relationships among the quantities in the problem	Giving an opportunity to the students to articulate and the represent all the quantities and relationships among the quantities, Giving an opportunity to students develop a representation (physical or visual, etc.) of the situation in order that students are modeling that comprise of all the quantities and their interrelationships Asking questions about a problem that focus on coordinating the dependence of one quantity on another quantity, coordinating the direction of change of one quantity with changes in the other quantity, and coordinating the amount of change of one quantity with changes in the other quantity (i.e. quantitative coordination).
Leading the students to execute calculations with reference to the quantities in the model or plan	Leading the students to describe calculations in terms of the quantities of the situation and relationships between these quantities. Leading the students to determine and interpret a meaning of a calculated value and how the value is associated with the purpose of the problem.
Leading the students to check the solution with respect to the quantities in the model or plan	Leading the students to check and justify the solution by providing them an opportunity to represent, interpret or coordinate the quantities and interrelationships, Leading the students about whether he could use different representations for the quantities or not, Leading the students to represent relationships among the quantities in a different way.

Results

The Quantitative Reasoning of the Prospective Teachers and Their Support for This Skill

Findings from the study revealed that five out of nine participants used their quantitative reasoning effectively in the problem-solving process. The participants with strong quantitative reasoning used a problem-solving approach that focused on the quantity in the problem-solving process via showing a behavior consistent with the problem-solving actions given in Table 1. In order to understand the context of the problem, these participants identified and interpreted the quantities in the problem by constructing a mental model (e.g. drawing a figure of the situation). Subsequently, they devised a plan in terms of quantitative meaning, structure and relations, and carried out this plan. As can be seen in the excerpt below, P1, who was one of the participants with strong quantitative reasoning, made sense of the problem by clearly stating the meanings and units of the quantities.

Interviewer: What does the problem tell you?
P1: The size of the sheet of paper measures 25 x 40 cm. In the present case, one side of the sheet of paper is 40 cm and the other side of the sheet of the paper is 25 cm. First of all, I want to visualize the model by drawing a figure. To put it simply, I want to draw a figure I imagined.
Interviewer: Of course.
P1: (drawing a figure of the sheet of paper and explains). Therefore, the length of a 25-cm x 40 cm sheet of paper is 40 cm and the width of a 25-cm x 40 cm sheet of paper is 25 cm. If I cut out equal-sized squares from each corner of this paper and fold the sides up, an open top box in the shape of rectangular prism will be formed.

These participants with strong quantitative reasoning determined relationships among the quantities and the changes in interrelationships among the quantities when devising a plan for solving these problems. Then they solved the problems by interpreting relationships among the quantities algebraically or geometrically, depending on the problem type. Furthermore, these participants justify the validity of the problem's solution in terms of relationships among the quantities in the problem situation. For example, P2 rearranged the formula she wrote that is relevant to the solution of the problem by considering the domain of quantities in the context of the problem.

P2: The height of the box formed by cutting off and folding up the corners of the paperboard is equal to the length of the side of each little square that was cut out. Let the length of the side of each little square that was cut out be x cm since how many centimeters of the length of the cutout is not given in the problem statement. In this case, the dimensions of the base of the box will be (25-2x) by (40-2x) cm. Because there are two cutout lengths on each side of the sheet of paper and so the base dimensions of the box decreased by (2x) cm resulted from cutting the paper when I cut out equal-sized squares from each corner of this paper. I wrote the function with respect to the volume of the box as ($V(x) = x\,(40-2x)\,(25-2x)$) since the volume of a rectangular prism is equal to the base area multiplied by the height.
Interviewer: What should the height of the box be in order to the box has the maximum possible volume?
P2: The box should be the maximum length concerning the base dimensions and its height in order to find the largest possible volume of the box. However, increasing in the height of the box resulted in decreases in the base dimensions of the box. Increasing in the base dimensions of the box caused decreases in the height of the box. Since the volume of a rectangular prism is equal to the base area multiplied by the height, it is necessary to increase the base dimensions and decrease the height of the box. Because increasing the length of the height of the box results in the two base dimensions to decrease in length. However, decreasing the length of the height of the box results in the two base dimensions to increase in length. Therefore, the height of the box should be a smaller value in order for the two base dimensions to get larger values in terms of the length. Thus, I can find the largest possible volume of the box.

In the solution process of the candle-burning problem, the thoughts and actions of P3 who was another participant with strong quantitative reasoning were as follows:

P3: There is 28 cm candle. This candle burns 3 cm per hour when lit. There are many variables in the problem.
Interviewer: What are the variables in the problem?
P3: In the given problem, time is a continuous changing variable. The unit of time is hour. The remaining length of the candle changes in terms of depending on time. I will write a function considering these variables. This formula was supposed to represent the remaining length of the candle with regard to the total amount of time the candle has burned.
Interviewer: Okay.
P3: There is a relationship between the time and the remaining length of the candle.
Interviewer: What kind of a relationship did you determine in this problem?
P3: The length of the candle decreased as time passed… Let's say, "t" represents time and the unit of time is hour. "x" represents the remaining length of the candle. The length of candle was 28 cm at the beginning. The length of candle decreased 3 cm per hour when lit. This meant that the length of the candle decreased three times as the time in terms of the unit of time. For instance; when a certain amount of time (t) passes, the candle burns 3t cm... The formula (writing $f(t) = 28 - 3t$) represents this relationship. This formula is a function determining the remaining length of the candle with respect to the total amount of time the candle has burned.

This study revealed that the participants with strong quantitative reasoning were able to support their middle school students' quantitative reasoning in the problem-solving process via a behavior consistent with the practical tips for teachers to help students develop quantitative reasoning in the problem-solving process given in Table 2. These participants primarily led their students to identify and interpret the quantities in the problem situation in order to enable them to understand the problem. Although their students wanted to solve the problem quickly, the participants led them to explore the meanings of the quantities in the problem situation rather than leading their student to solve the problem.

Furthermore, although students of these participants incoherently and incompletely expressed the quantities in the problems, they guide by asking questions to make students express the quantities meaningfully and appropriately. The guidance provided by the participants helped students to appropriately express the quantities and consider units of the quantities. In the following excerpt, it was seen that P4 led her student to explore the meaning of the quantities. P4 guide by asking questions to make students express the measurable attribute of the quantities, as can be seen in the excerpt below.

Student: I can solve this problem by using x and y.
P4: Okay. You can use any approach to solve this problem. However, I would like you to tell me the problem before you solve it.
Student: Flutes are three times as many as violins.
P4: I cannot understand. What are three times as many as what?
Student: The number of students who play the flute is three times as many as those who play the violin. The number of the students who play the guitar is 20 less than those who play the flute. There are to 127 students playing an instrument in this school. The number of students who play the guitar is asked.

These participants enabled their students to devise a plan in terms of quantitative relationships by leading their students to determine relationships among the quantities and to represent these relationships. Although some students tried to represent the quantities by writing an algebraic equation, they incoherently and incompletely explained which letter or symbol represents which quantity in the problem. Therefore, these participants led their students to interpret in detail the algebraic representation they used regarding the quantities. This guidance enabled students to consider the measurable attribute of an object associated with the quantity and to appropriately express the quantities when they identified the quantities in the problem.

Furthermore, these participants led their students to interpret in detail the algebraic equations they wrote by giving a justification in terms of relationships among the quantities in the problem. The following excerpt showed the student guidance of P5 with regard to quantitative meaning.

P5: Why do you think you could solve this problem with an equation?
Student: Because the number of big-size and small-size trucks was asked in the problem and it is not given in the problem statement. For this reason, I assign a letter to represent big-size and small-size trucks. (Writing). Let "x" represents big-size trucks and the "y" represents small-size trucks.
P5: I do not understand what you mean. You stated that the "x" represents big-size trucks and the "y" represents small-size trucks. What do you mean by this sentence?
Student: (Laughing). Since I do not know how many big-size and small-size trucks are in the problem statement, the "x" represents the number of big-size trucks and y" represents the number of small-size trucks…
P5: How did you write the algebraic equation (30x+25y =675)? …
Student: In this equation, the "x" represents the number of big-size trucks and the number of beds carried by each big-size truck per trip was 30. In a similar way, the "y" represents the number of small-size trucks and the number of beds carried by each small-size truck per trip was 25. Therefore, I wrote the equation (30x+25y=675).

The questioning of P1 regarding another problem is seen in the excerpt below:

P1: How did you write the algebraic equation (127=7x-20)?
Student: The total number of students who play an instrument is 127. These students play only one out of three instruments. The "x" represents the number of students who play the violin in the equation I wrote. I focus on the relationships in the problem situation. In this equation, the number of students who play the flute and the guitar represent 3x and 3x-20, respectively. I added them all

up and got x + 3x + 3x - 20 = 7x - 20. Since the number of students who play an instrument in this school is 127, I wrote the equation (7x-20=127).

P1: What does this equation, (7x-20=127), mean?

Student: This equation means that 20 less than seven times the number of students who play the violin is equal to 127 people.

On the other hand, the problem-solving actions given in Table 1 was quite rarely observed in the interviews conducted by the researchers with four prospective teachers who have poor quantitative-reasoning. However, these participants attempted to perform the prescribed behavior and goal (outcome) oriented problem-solving behavior which a problem solver focuses on numbers and operations or uses memorized rules and formulas in the problem. These prospective teachers tried to solve the problems incompletely in terms of quantitative meaning.

The participants executed calculations and use formulas and procedures rather than spending time describing and analyzing the context of the problem. These prospective teachers did not analyze the formulas they used in terms of the quantitative relationships in the context of the problem or were unable to recall the formulas in order to use them in a meaningful way. For example, the following excerpt indicates that P6 wanted to solve the problems by using solution approaches of similar problems that had already been solved in the past.

Interviewer: What should the height of the box be in order to the box has the maximum possible volume?

P6: The derivative comes to my mind. I remember using derivatives when solving this type of problems. Because we used derivatives to solve maximum-minimum problems.

Interviewer: Why you solved such problem by using derivatives?

P6: I could not remember how and why we use derivatives to solve such problems. Did the graph of derivative function change its direction at a point where the derivative of the function equals to 0? (Silence) I am confused. I cannot solve the problem.

In the same vein, the statements of P7, who recalled the procedures or the approach of solution without conceiving of the box-problem situation, were as follows:

P7: Yes. In order to find the largest possible volume of the box, the side of a square should be the greatest common divisor of 25 and 40.

Interviewer: Why should the length of the side of each little square be the greatest common divisor of 25 and 40?

P7: The facility in performing operations is provided. In order to solve these types of problems, the greatest common divisor is found. The squares that was cut out should completely cover the box. Therefore, the squares should be a factor of each side of the paper.

It was seen that the participants with poor quantitative reasoning incoherently and incompletely expressed or used the quantities in the problems. The participants usually expressed the quantities in the problems by the name of object or phenomenon regarding the quantities. Although these prospective teachers represent the quantities in the context of the problem by assigning a letter, they did not identify the units of the quantities. These participants could not algebraically or geometrically interpret relationships among the quantities and the changes in the quantities since they tried to solve the problem by using memorized algorithms, rules and formulas.

For example, all of these participants consider the quantity of time as a discrete variable in the candle-burning problem and they did not analyze the change of the time variable. Therefore, these participants could not state that the letter or the symbol represents which quantity in the algebraic formulas they wrote. For instance, the problem-solving actions of P7, P8 and P9 regarding the candle-burning problem are seen in the excerpt below:

P8: It burns 3 cm in an hour. I assign "x" to represent the burning of the candle for a certain period of time. Therefore, "28-x" give the remaining of the candle. Since it burns 3 cm per hour, the formula would be 28-x=3t. In this formula, both "x" and "t" represents the hour. The symbol of "x" represents the amount of melted candle, the amount of the remaining of the candle and the time…

Interviewer: What is this formula supposed to mean?

P8: This formula should determine the remaining length of the candle with respect to the total amount of time the candle has burned. The formula I wrote determines how many hours the candle will burn. No, it did not work out. If the candle burns 3 in an hour, I write a proportion that gives how many hours it takes to burn "28-x". I find this equation (28-x = 3t). Therefore, I found that how much time the remaining of the candle will burn completely. I could not do it again (silence). Let's assign "t" to represent the time. If the candle burns 3 in an hour, I write a proportion that gives how many hours it takes to burn "t". In "t" hour, the candle burns "3t". Thus, I found the formula that was asked in the problem as 3t = 28 - x…

P9: The candle burns 3 cm per hour. The length of the candle varies by time. If we assign "a" to represent the length, then a → t forms.

Interviewer: What does "a → t" mean?

P9: The arrow indicates that "a" is dependent on "t". Derivatives were used to solve the problems that includes the variations in terms of depending on time. The initial length of the candle was 28. I need to think the instantaneous velocity since I should consider the time from t to t0. The instantaneous velocity implies the derivative... I have to write a derivative regarding the candle by using limits. If I correctly remember the instantaneous velocity, this formula $\lim_{t \to t_0} \frac{28-(28-3t_0)}{t-t_0}$ represents the remaining of the candle…

P7: The candle burns 3 cm in an hour. The length of the candle is 28 cm. If I use this formula (X = V.t), I will write a formula that was asked in the problem.

Interviewer: What does "X = V.t" mean?

P7: This formula expresses the change of length (x) in terms of depending on time. The length of the candle (x) changes as depending on time (t). However, I have no idea how to use the variable of speed which represents "V" in the problem.

On the other hand, it was seen that the participants, who have poor quantitative-reasoning, could not demonstrate the ways of guiding a middle school student in the problem-solving process consistent with the recommendations for teachers to support students develop quantitative reasoning given in Table 2. The study revealed that these participants were unable to lead their students to support their quantitative reasoning. These prospective teachers led their students to focus on numbers in the problem situation and to perform a calculation by ignoring the quantities in the problem. It was seen that these participants led their students to perform a calculation in order to enable them to solve the problem during the clinical interviews conducted by the prospective teachers with one middle school student, rather than leading their student to identify and interpret the quantities in the context of the problem in order to enable them to understand the problem.

The most remarkable point in the guidance of these participants was that the students of these participants incoherently, incompletely and inaccurately expressed or used the quantities in the problems although they did not guide by asking questions to make the students realize their mistakes with regard to inappropriate expressions of the quantities. Furthermore, the participants usually expressed the quantities in the context of the problem by the name of object or phenomenon during the clinical interviews. Moreover, it was not clear which quantity or which measurable attributed to the quantity were stated or expressed by these participants when they asked question to the middle school student.

In addition to this, the most remarkable characteristic of the interview conducted by P9 was that even though the student answered about quantities that were irrelevant to P9's questions, she kept asking questions to the student without realizing it.

P9: How many classrooms are there at 5th grade?
Student: There are 72 students at 5th grade in this school.
P9: How many students are there at 6th grade?
Student: There should be two 6th grade classrooms.
P9: Yes, the 5th grade is 72, isn't it?
Student: (silence) okay.
P9: If it is more than 8, how can you find 6th grade?

Among the participants, P6, P7, P8 and P9 asked students to identify the relationships in the problem without identifying the quantities. Their students could not realize whether there was a relationship between the quantities in the problem since they did not identify the quantities in the problem. Therefore, these participants' leading their students to determine relationships among the quantities caused their students to get confused.

P9: What kind of relationship is there between 5th and 6th grades?
Student: I could not understand (silence).

P9: (reading the problem again). Now, if 72 students were at 5th grade in a school, which was 8 more than were in the 6th grade, then what kind of connection is there between 5th and 6th grades?
Student: (long silence). I could not see a connection in the problem and I could not understand…

P6, P8 and P9 led their students, who were unable to comprehend the problem in terms of quantitative meaning, to perform a calculation in order to enable them to find the solution of the problem. For example, P6 led the student to use the strategy of look for a keyword and perform an operation even though the student could not recognize the relationship between the quantities in the problem or even understand the problem. The student of P6 performed meaningless calculations, got confused and took a deep breath since he could not understand the problem.

P6: All right, when does Jale become three times as Sinem?
Student: 12 years later.
P6: Yes, how do you solve this problem? What do we usually do to find the age after 12 years?
Student: Hmm! We perform addition. I add 153 to 12 and it equals 165 (silence). I cannot understand the problem. I am confused.
P6: Read the problem again... Let's perform different operations now.
Student: (reading the problem) … I still cannot understand the problem and I am confused again (taking a deep breath).

The most remarkable characteristic of the interview conducted by P9 was that there was the difference between P9's and her student's ways of expressing the quantities. Although the student of P9 considered the measurable attribute of an object associated with the quantities and the unit of the quantities when expressing the quantities, P9 handled the object itself as a quantity and P9 did not realize that her student meaningfully and accurately expressed the quantities during the interview. To start with, student of P9 performed meaningless calculations, and then he did not answer the questions of P9 for a long time, then tried to understand the problem quietly alone. Ultimately, the student of P9 was able to solve the problem on his own. After the student solved the problem, he justified the calculations he performed in terms of quantitative meaning even though P9 did not ask any questions to him.

P9: What kind of connection is there between flute and violin?
Student: (silence) I could not understand. I could not see any connection...
P9: (reading the problem). Now, how can you write a connection between guitar and flute mathematically?
Student: Let me read the problem on my own. (reading the problem) Hmm. The number of the students who play the guitar is 20 less than those who play the flute. The number of students who play the guitar in this school is asked.
P9: How can we find those who play the guitar?
Student: How can we find them? (silence). I cannot understand the problem.? …May I read it again?...
P9: You solved the problem on your own. Now, how did you represent the violin?
Student: Since the number of the students who play the violin is not given in the problem statement, I assign "k" to represent the number of students who play the violin.
P9: How did you represent the flute?
Student: The number of students who play the flute is three times as many as those who play the violin. Therefore, if I assign "k" to represent the number of students who play the violin, then I should assign "3k" to represent the number of students who play the flute.

In conclusion, the study revealed that the participants with strong quantitative reasoning and the participants with poor quantitative reasoning exhibited problem-solving behaviors different from each other, and furthermore, their support for middle school students' quantitative reasoning in the problem-solving process was parallel to their own quantitative reasoning. In order to look at the results in the context of the research from a broad perspective, the participants' problem-solving actions and their ways of guiding middle-school students were given in Table 3.

Table 3. Participants' problem-solving actions and ways of guiding students

Participants Dispositions	Participants with strong quantitative reasoning	Participants with poor quantitative reasoning
Problem-solving actions	Problem-solving behavior associated with quantitative reasoning - First of all, attempting to understand the context of the problem, - Focusing on the quantities in the context of the problem, - Identifying the quantities in the context of the problem, - Meaningfully expressing the quantities in the context of the problem, - Determining relationships among the quantities or analyzing the changes in the quantities in the context of the problem, - Devising a plan in terms of quantitative relationships or the changes in the quantities, - Executing a calculation focusing on the quantities and relationships among the quantities in a devised plan, - Obtaining the solution of the problem, - Justifying the validity of the solution in terms of quantitative meaning, - Exhibiting confidence in the solution s/he obtained.	Prescribed behavior and goal (outcome) oriented problem-solving behavior which a problem solver focuses on numbers and operations or uses memorized rules and formulas in the problem - Interpreting the context of the problem by focusing on keywords, -Focusing on numbers in the context of the problem, -Incompletely or incoherently expressing the quantities in the context of the problem, - Recalling a previously solved problem that is similar to current problem, - Recalling and using a procedure/formula without devising a plan in terms of quantitative meaning, - Trying to recall the previous procedural operations and steps of a rule/formula, - Executing calculations without focusing on the quantities and relationships among the quantities, - Failing to obtain the solution of the problem, - Doing not anything to justify the validity of the problem's solution, - Doubting the correctness of the solution s/he obtained.
The ways of guiding middle-school students	Student guidance in a manner that focuses on quantitative reasoning - Leading the students to identify and interpret the quantities in the problem situation in order to enable them to understand the context of the problem, - Asking questions to make students express the quantities meaningfully and appropriately in the context of the problem, - Leading the students to determine relationships among the quantities or to analyze the changes in the quantities and to represent them in the context of the problem, - Leading the students to devise a plan in terms of quantitative relationships or the changes in the quantities in the context of the problem, - Leading the students to execute calculations with respect to the quantities in a devised plan, - Leading the students to check the solution with respect to the quantitative relationships in a devised plan, * Guiding the interview by questioning step-by-step in terms of the components of quantitative reasoning.	Student guidance in a prescribed and goal-oriented manner - Leading the students focus on numbers in the context of the problem in order to solve the problem quickly, - Leading the students look for keywords in the context of the problem in order to solve the problem quickly, - Asking questions about a problem by incoherently and incompletely expressed the quantities in the problem, - Leading the students who do not identify the quantities in the context of the problem and do not understand the problem to determine the quantitative relationships, - Leading the students who do not devise a plan in the context of the problem to execute calculations or use algebraic manipulations, - Leading the students to recall the procedural operations with respect to the solution of the current problem type or to give a hint, -Not needing to lead the students to justify the validity of the problem's solution, * Guiding the interview by questioning that focuses on calculation and solution.

Prospective Teachers' Conceptions in the Context of Quantitative Reasoning and Developing Quantitative Reasoning

According to the results of the study, all prospective teachers reflected that they had an idea about quantity, quantitative reasoning and developing middle-school students' quantitative reasoning. All of these prospective teachers demonstrated that they had more or less information about the concepts of quantity and quantitative reasoning. In addition, it was seen that the prospective teachers' conceptions of these concepts and their ways of developing middle-school students' quantitative reasoning differed in terms of their own quantitative reasoning. The interviews conducted by the researchers with the participants revealed that the participants with strong quantitative reasoning tried to more accurately identify and express these concepts as compared to the participants with poor quantitative reasoning. These participants attempted to concrete quantity by giving examples and to meaningfully clarify the difference the number and the quantity. For example, P5 emphasized that the quantity consists of the object, the measurable attribute of the object or phenomenon, and the unit of measure of this attribute. Furthermore, P5 described the quantity as a process that results from assigning a numerical value to the measurable attribute of an object.

Interviewer: What is the quantity?

P5: Any quantity should have a measurable attribute and a unit. Since any quantity has a measurable unit, we can assign numbers to measurable qualities by considering the unit of the quantity. Therefore, any quantity should have a measurable unit

Interviewer: What do you think about number and quantity?

P5: The quantity results from assigning a numerical value to the measurable attribute of an object or a thing. I can explain the difference between quantity and number by giving an example. For example, if we only say "5", then we need to know what "5" represents. We should consider that if we mean 5 apples or 5 pears. For instance, if "5" is assigned to represent the height of a tree then it should be expressed as 5 meters. Expressing only "5" verbally or in writing does not create anything in my mind. The difference between quantity and number is that any quantity should consist of an object, a measurable attribute and a process by which an individual assigns a number to measurable qualities by considering the unit of the quantity. Therefore, quantity is more complex than number.

The participants with strong quantitative reasoning identified most of the components of quantitative reasoning when defining the concept of quantitative reasoning. These participants stated that the algebraic-verbal problems can be solved by using these components.

Interviewer: What is quantitative reasoning?

P2: Quantitative reasoning is everything that is done to find what is asked for in the problem depending on the quantities by focusing the measurable qualities of the givens in the problem, using and interpreting the quantities and determining relationships among the quantities. As a matter of fact, middle-school mathematics problems can only be solved by understanding and interpreting the quantities, and determining relationships among the quantities. Therefore, a middle school student can solve the problem by understanding the quantities and relationships among the quantities, and using them.

P3: Quantitative reasoning is the ability to see and understand everything that is given quantitatively, as well as to determine and interpret relationships among the quantities. Quantitative reasoning is to determine relationships among the quantities in the problem, think about how quantities are represented in the problem and imagine the changes in the quantities.

The participants with strong quantitative reasoning indicated that they lead their students to use the skills such as identifying the quantities in the problem situation, representing the quantities, determining relationships among the quantities, coordinating appropriate units of the quantities that are different from each other (quantitative unit coordination), determining the changes in interrelationships among the quantities and representing them algebraically.

Interviewer: How do you lead your students to develop their quantitative reasoning when solving an algebraic verbal problem?

P1: First of all, in order to make my students to understand the quantities in the problem, I discuss with my students what quantities are given in the problem, what are the units of the quantities, what quantities are known, what quantities are unknown and so on. Then, I enable my students to explore relationships among the quantities in the problem statement. If my students are in the early stages of the transition from arithmetic to algebra, then I would ask them to generalize relationships among the quantities in the problem by drawing a box, a diagram or a figure. Furthermore, I would ask my students who are the later stages of the transition from arithmetic to algebra to generalize relationships by writing an algebraic equation using the letters. I would also ask my students to interpret a quantity in the problem in terms of another quantity and to represent it algebraically.

P4: I lead my students to make a table in order to make my students explore the relationships in the problem. I lead my students to represent relationships among the quantities by using various representations such as box, diagram etc. I lead them to determine the changes in interrelationships among the quantities in order to make my students to imagine the changes in the quantities. If the units of the quantities are different, I ask my students what they should think about coordinating appropriate units of the quantities that are different from each other.

The debriefing interviews revealed that the participants with strong quantitative reasoning had strong conceptions regarding the quantity, the quantitative reasoning and developing quantitative reasoning. It was also seen that these prospective teachers purposefully and effectively led their students in the context of supporting middle-school students' quantitative reasoning, during the interviews conducted by them with middle-school students. In other words, the participants put their conceptions of the quantitative concepts into practice both when solving the problems with the researcher and when leading the middle-school student to solve the problems. On the other hand, the participants with poor quantitative reasoning incoherently, incompletely and inaccurately expressed or used these quantitative concepts. These participants stated the concept of quantity as only numerical value/data, object or measurable attribute. None of these prospective teachers described the quantity as a process as well as they incompletely identify the features that constitute the quantity. Moreover, it was revealed that P6 and P7 had a wrong conception of the quantity concept since the quantity was defined as a number, value or numerical data by these participants.

Interviewer: What is the quantity?

P6: Numerical data that are given in the problem are quantity.

P7: Quantities are numbers/values that are given in the problem. For example, the number "5" in five apples is a quantity. The "apple" is also a quantity. The shape of an apple is another quantity.

P8: I cannot define the quantity but…(laughing). For example, people who play the flute and the violin and the students who have colored-eyes in the classroom are quantities.

The study revealed that these participants considered quantitative reasoning as only determining relationships among the quantities in the problem. None of the participants described quantitative reasoning as a skill consisting of components such as identifying, representing and interpreting the quantities and determining relationships among the quantities. Furthermore, the interview showed that P6 had a wrong conception of quantitative reasoning since she considered quantitative reasoning as comprehending the relationships between numerical values/data. Although the participants with poor quantitative reasoning stated that they would only lead their students to determine the relationship among the quantities in problem situation in order to develop their quantitative reasoning skills, they did not explain how they would lead their students to determine the relationship among the quantities in problem. For example, P8 pointed out that she used representations and models to make their students determine relationships among the quantities in the problem situation but did not explain how to use concrete tools for this purpose.

Interviewer: How do you lead your students to develop their quantitative reasoning when solving an algebraic verbal problem?

P8: I enable my students to understand relationship among the quantities in the problem by drawing figures, tables and diagrams, and using bar charts and bar models.

P7: It is important for students to recognize what number is more or less than what number and that there is a certain relationship between numbers in the problem. In short, my students should recognize that the numbers in the problem are not randomly given. Therefore, I lead my students by focusing on the numbers and relationships among the numbers in the problem.

The debriefing interviews revealed that, contrary to the participants with strong quantitative reasoning, the participants with poor quantitative reasoning had poor conceptions regarding the quantity, the quantitative reasoning and developing quantitative reasoning in comparison with the participants with strong quantitative reasoning. It was seen that the conceptions of these participants about these concepts and developing middle-school students' quantitative reasoning were incomplete, incoherent or inaccurate, and were not appropriate for terminological structure. At the same time, these prospective teachers neither supported their students' quantitative reasoning nor guide their students purposefully in terms of reasoning quantitatively during the interviews conducted by them with middle-school students. Moreover, the participants asked questions that hinder their students to reason quantitatively in the interviews. Since the conceptions of these participants about quantitative concepts were incomplete, they reflected incomplete conceptions of these concepts both when solving the problems with the researcher and when leading the middle-school student to solve the problems.

Discussion and Conclusion

The results of the study indicated that prospective teachers with strong quantitative reasoning and prospective teachers with poor quantitative reasoning exhibited problem-solving behaviors different from each other. It was observed that the participants with strong quantitative exhibited problem-solving behaviors consistent with the features of the quantitative reasoning given in Table 1, whereas the participants with poor quantitative reasoning had rarely exhibited problem-solving behaviors regarding the features of the quantitative reasoning in the problem-solving process. The use of a problem-solving approach that focused on the quantity by prospective teachers with strong quantitative reasoning caused them to determine relationships among the quantities in the problem and to conceive the context of problem in terms of quantitative meaning. Thus, this approach enabled them to conceive the problems in an effective way, to algebraically represent relationships among the quantities and to solve the problems in a productive way. On the other hand, the use of a problem-solving approach that focused on performing calculations devoid of quantitative meaning without understanding the problem caused them unable to identify the quantities, to determine relationships among the quantities in the context of the problem and to conceive the context of problem in terms of quantitative meaning. Thus, this approach hindered them to solve these problems. This result provides further support for Mayer, Lewis and Hegarty's (1992) claim that the main difficulty in solving the problems is firstly performing calculations rather than constructing a cognitive model based on quantitative relationships in the problem situation. Many studies in the literature (e.g. Ellis, 2007; Moore, 2011) that quantitative reasoning play a key role in problem-solving process and this skill ensures solving the problems in an efficient way.

The results of the study revealed that prospective middle-school mathematics teachers' ways of supporting middle-school students' quantitative reasoning in the problem-solving process were parallel to their own quantitative reasoning. It was seen that the participants with strong quantitative reasoning were able to use appropriate questioning in terms of the recommendations for teachers to support students develop quantitative reasoning given in Table 2, whereas questioning of the participants with poor quantitative-reasoning was not consistent with these recommendations. The prospective teachers with poor quantitative reasoning led their students to perform calculations and operations, while they did not support their students' skills in terms of constructing the quantities and determining relationships among the quantities. As Moore (2011) emphasized, the guidance of these participants which was deficient in respect of quantitative meaning and focused on performing calculations/operations did not assist their middle-school students in solving algebraic-verbal problems during the clinical interviews. Sowder (1988) qualified a learning environment which focuses on numbers and operations instead of quantities and relationships among the quantities in problem-solving process as an ungrounded and useless discussion. Considering in the context of this qualification by Sowder, as long as the quantitative reasoning skills of prospective teachers with poor quantitative reasoning are not developed enough and their support for this skill are not provided enough, the learning environments that these teachers create for students may also be as a useless discussion. Although quantitatively rich problems were used in clinical interviews, these problems did not support prospective teachers with poor quantitative reasoning to focus on the quantities and to lead their students to the quantity. This result supports the view of Ellis (2007) who suggests the solving quantitatively rich problems does not imply that it will improve the problem-solving skills and quantitative reasoning skills of the students. Moreover, Ellis (2011) points out that solving quantitatively rich problems in learning environment does not guarantee students' quantitative reasoning and will not function as a panacea for all problems in solving the problem. As Ellis (2007) stresses, supporting middle-school students' quantitative reasoning is only possible with middle-school mathematics teachers who are able to use well-structured questioning. In order to prepare students better to solve algebraic-verbal problems, mathematics teachers should engage students' quantitative reasoning and intend to develop an understanding based on the quantities and relationships among the quantities in the learning environment (Moore, 2011).

One of the most important results of this study is that there is a strong relationship between prospective middle-school mathematics teachers' quantitative reasoning and their support for students' quantitative reasoning in the problem-solving process. This result is consistent with Van Den Kieboom's et al. (2014) result that prospective mathematics teachers with higher algebraic reasoning were able to use appropriate questioning to foster students' algebraic reasoning, whereas prospective mathematics teachers with lower algebraic reasoning were unable to use appropriate questioning to foster students' algebraic reasoning. In this study, prospective middle-school mathematics teachers with poor quantitative reasoning were not able to support their middle-school students' quantitative reasoning in the problem-solving process. A one semester elective course seems not to be enough for the development of these prospective teachers' quantitative reasoning and their support for students' quantitative reasoning in the problem-solving process. The reason for this situation is that teachers' sufficient content knowledge of mathematics and skills are essential for enhancing their pedagogical content knowledge, as Baumert and colleagues (2010) state. It is thought that this result may arise from a relationship between insufficient mathematical content knowledge and skills, and the lack of pedagogical content knowledge (Van den Kieboom et al., 2014). It is seen that the elective course in which the study is conducted remains weak for developing quantitative reasoning of prospective teachers with poor quantitative reasoning and supporting their pedagogical approaches in relation with this skill. As a matter of fact, the debriefing interviews also showed that these prospective teachers had awareness of students' quantitative reasoning at knowledge level but could not make sense of the concepts in this context. Since prospective middle-school mathematics teachers with strong quantitative reasoning had sophisticated quantitative reasoning before taking an elective course in relation with the quantitative reasoning and its pedagogy, this might have also helped them to provide awareness in quantitative reasoning and its teaching that allows them to put their knowledge into practice. Considering a fact emphasized in the literature that a year or two of mathematical content courses and pedagogy courses are not enough to promote prospective teachers' quantitative reasoning (Smith & Thompson, 2008), it is thought that prospective teachers with strong quantitative reasoning have gained these skills throughout the mathematics learning process before taking the elective course. However, it seems that even a one-semester course in the pedagogical context that aims to support students' or teachers' quantitative reasoning is quite efficient for a prospective teacher who has sophisticated quantitative reasoning. As Van Den Kieboom et al. (2014) emphasizes, the compulsory courses are necessary for developing prospective teachers' pedagogical approaches in this context. Quantitative reasoning should be constantly and continuously promoted in the teaching of primary, middle and secondary school mathematics. Moreover, the mathematics teacher education curriculum should enable prospective teachers who have strong quantitative reasoning to acquire qualifications, in terms of both in terms of both content knowledge and pedagogical content knowledge. In order to develop quantitative reasoning skills of students, prospective teachers and teachers, all school mathematics curricula, textbooks and mathematics teacher training should be strengthened in all aspects. Several studies (e.g. Post, Harel, Lesh, & Behr, 1991; Stigler, Fuson, Ham, & Kim, 1986) reveal that many mathematics teachers who use the textbooks and curricula do not support quantitative reasoning appropriately, find providing support for middle school students' quantitative reasoning surprising, and most of them cannot use quantitative reasoning effectively.

A further long-term study using teaching experiment methodology, which aims to develop participants' quantitative reasoning and their support for this skill, may focus on the relationship between prospective middle-school mathematics teachers' quantitative reasoning and their support for middle-school mathematics students' quantitative reasoning in the learning environment. Furthermore, the results of the study indicate that the frameworks which was adapted from the studies of Moore (2011) and Weber et al. (2014) can be used to assess both the problem-solving process and the teacher's guidance in terms of quantitative reasoning. Table 3 that characterizes the participants', who have poor or strong quantitative reasoning problem-solving actions and their ways of guiding middle-school students, can also be used to analyze the data in future studies.

Acknowledgements

This research was supported by Anadolu University Scientific Research Projects under Grant No. 1404E273.

References

Akkan, Y., Baki, A., & Çakıroğlu, Ü. (2012). 5-8. Sınıf öğrencilerinin aritmetikten cebire geçiş problem çözme bağlamında incelenmesi. *Hacettepe Eğitim Fakültesi Dergisi, 43*, 1-13.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., … & Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47(*1), 133–180.

Cai, J., & Knuth, E. (2011). *Early algebraization: A global dialogue from multiple perspectives.* Springer-Berlin Heidelberg.

Cai, J., & Nie, B. (2007). Problem solving in Chinese mathematics education: Research and practice. *ZDM, 39*(5-6), 459-473.
Carlson, M. P. (2013). Supporting student success in developing meaningful formulas in word problems by conceptualizing quantities and how they co-vary. *Oncore* (pp. 29-34). AATM-Arizona Association of Teachers of Mathematics, Arizona, USA.
Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45-75.
Clement, J. (2000). Analysis of clinical interviews: Foundation and model viability. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). New Jersey: Lawrence Erlbaum.
Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction, 25*(4), 439–478.
Ellis, A. B. (2009). Patterns, quantities, and linear functions. *Mathematics Teaching in the Middle School, 14*(8), 482-491.
Fraenkel, J. R., & Wallen, N. E. (1996). *How to design and evaluate research in education* (Third Edition). New York: McGraw-Hill.
Goe, L. (2007). *The link between teacher quality and student outcomes: A research synthesis*. Washington, DC: National Comprehensive Center for Teacher Quality.
Goldin, G. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). New Jersey: Lawrence Erlbaum.
Harel, G. (2007). The DNR System as a conceptual framework for curriculum development and instruction. In R. Lesh & J. Kaput, E. Hamilton (Eds.), *Foundations for the future in mathematics education* (pp. 263-280). Mahwah, NJ: Erlbaum.
Kaput, J. (1995). Long term algebra reform: Democratizing access to big ideas. In C. Lacampagne, W. Blair & J. Kaput (Eds.), *The Algebra initiative colloquium* (pp. 33–52). Washington, DC: U.S. Department of Education.
Koichu, B., & Harel, G. (2007). Triadic interaction in clinical task-based interviews with mathematics teachers. *Educational Studies in Mathematics, 65*(3), 349-365.
Mayer, R. E., Lewis, A. B., & Hegarty, M. (1992). Mathematical misunderstandings: Qualitative reasoning about quantitative problems. *Advances in Psychology, 91*, 137-153.
McAlpine, L., Weston, C., & Beauchamp, C. (2002). Debriefing interview and colloquium: How effective are these as research strategies? *Instructional Science, 30*(5), 403-432.
Miles, M. B., & Huberman, A. M. (1994). *An expanded sourcebook qualitative data analysis* (Second Edition). Thousand Oaks, California: Sage Publications.
Moore, K. C. (2010). *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry*. Unpublished Dissertation. Arizona State University, USA.
Moore, K. C. (2011). Relationships between quantitative reasoning and students' problem-solving behaviors. In S. Brown, S. Larsen, K. Marrongelle & M. Oehtman (Eds.), *Proceeding of the 14th Conference on Research in Undergraduate Mathematics Education* (Vol. 4, pp. 298-313). Portland, OR: Portland State University.
Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *The Journal of Mathematical Behavior, 31*(1), 48-59.
Moore, K. C., Carlson, M. P., & Oehrtman, M. (2009). The role of quantitative reasoning in solving applied precalculus problems. *Proceedings of the Twelfth Annual Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC: North Carolina State University.
Moyer, P. S., & Milewicz, E. (2002). Learning to question: Categories of questioning used by preservice teachers during diagnostic mathematics interviews. *Journal of Mathematics Teacher Education, 5*(4), 293-315.
Olive, J., & Çağlayan, G. (2008). Learners' difficulties with quantitative units in algebraic word problems and the teacher's interpretation of those difficulties. *International Journal of Science and Mathematics Education, 6*(2), 269-292.
Polya, G. (1957). *How to solve it: A new aspect of mathematical methods* (2nd ed.). Garden City, NJ: Doubleday.
Post, T. R., Harel, G., Behr, M., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. Carpenter & S. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177 – 198), SUNY, Albany, NY State University of New York Press.
Rigelman, N. R. (2007). Fostering mathematical thinking and problem-solving: The teacher's role. *Teaching Children Mathematics, 13*(6), 308-314.

Schoenfeld, A. H. (2007). Problem solving in the United States, 1970-2008: research and theory, practice and politics. *ZDM Mathematics Education, 39*(5-6), 537-551.

Smith, J., & Thompson, P. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput & D. Carraher (Eds.), *Algebra in the early grades* (pp. 95-132). New York, NY: Lawrence Erlbaum Associates.

Sowder, L. (1988). Children's solutions of story problems. *Journal of Mathematical Behavior, 7*, 227-238.

Steffe, L. P., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.*), Research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Lawrence Erlbaum Associates.

Stigler, J. W., Fuson, K. C., Ham, M., & Sook Kim, M. (1986). An analysis of addition and subtraction word problems in American and Soviet elementary mathematics textbooks. *Cognition and Instruction*, *3*(3), 153-171.

Strauss, A. L., & Corbin, J. M. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (2nd Edition). Thousand Oaks: Sage Publications.

Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glaserfeld* (pp. 412–448). London: Falmer Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education. WISDOMe Mongraphs* (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.

Van Den Kieboom, L. A., Magiera, M. T., & Moyer, J. C. (2014). Exploring the relationship between K-8 prospective teachers' algebraic thinking proficiency and the questions they pose during diagnostic algebraic thinking interviews. *Journal of Mathematics Teacher Education*, *17*(5), 429-461.

Watson, A., & Harel, G. (2013). The role of teachers' knowledge of functions in their teaching: A conceptual approach with illustrations from two cases. *Canadian Journal of Science, Mathematics and Technology Education, 13*(2), 154-168.

Weber, E., Ellis, A., Kulow, T., & Ozgur, Z. (2014). Six principles for quantitative reasoning and modeling. *Mathematics Teacher, 108*(1), 24-30.

Author Information

Tangul Kabael
Anadolu University
Faculty of Education, Department of Mathematics and Science Education, Eskişehir, 26470, Turkey
Contact e-mail: *tuygur@anadolu.edu.tr*

Ayca Akin
Kastamonu University
Faculty of Education, Department of Mathematics and Science Education, Kastamonu, 37200, Turkey

Appendix

The problems used in the exploratory-teaching interviews with prospective-middle school mathematics teachers.

Problem 1 (Box problem): Starting with a 25-cm x 40 cm sheet of paper, a box with an open top is formed by cutting equal-sized squares from each corner of the paper and folding the sides up. Write a formula regarding the volume of the box that was formed. What size squares should be cut to obtain the largest possible volume of the box? (adapted from Moore and Carlson (2012)).

Problem 2 (Candle burning problem): Melisa bought a 28-cm candle which burns 3 cm per hour when lit. Since Melisa wants to use this candle at a ceremony, she needs a model that demonstrates how the length of the candle changes while burning. Write a formula that Melisa can use. This formula should determine the remaining length of the candle with respect to the total amount of time the candle has burned (adapted from Carlson (2013)).

Problem 3 (Coin problem): Selin has some coins consisting of 5 kurus, 10 kurus and 25 kurus. The number of 10 kurus is three more than the number of 5 kurus, the number of 25 kurus is two less than the number of 5 kurus. Since the total value of the coins is 9 liras and 40 kurus that Selin has, how many 5-kurus, 10-kurus and 25-kurus coins does she have? What should you pay attention to when teaching this problem? How would you organize the solution process of this problem? (adapted from Olive and Çağlayan (2008)).

The problems used in the clinical interviews conducted by the prospective teachers with one middle school student.

Problem 1: 12 years later, Jale's age will be 3 times more than Sinem's age. If Jale's present age is 51, then how old is Sinem now?

Problem 2: A middle school has two 5th grade classes, Room 1 and 3, and two 6th grade classes, Room 2 and 4. There are 72 students at 5th grade in this school, which is 8 more than are in the 6th grade. If the total number of students in Room 2 is 34, then how many students are in Room 4? (adapted from Smith and Thompson (2008)).

Problem 3: 127 students in a middle school play either the flute, the guitar or the violin. The number of students who play the flute is three times as many as those who play the violin. The number of the students who play the guitar is 20 less than those who play the flute. How many students play the guitar in this school?

Problem 4: A furniture factory transports 675 beds from Turkey to Germany using small-size and large-size trucks. A big truck carries 30 beds per trip, and a small truck carries 25 beds per trip. When trucks were carrying the beds, it is seen that there were 5 more small trucks than big trucks in the convoy. How many big-size and small-size trucks were in this convoy?

Problem 5: A company transports wheat using one small size truck and one large size truck. The total weight of wheat was 69 tons. The large-size truck carries 7 tons of wheat per trip, and the small-size truck carries 3 tons of wheat per trip. In total, the trucks made 15 trips to carry all wheat. How many trips did each truck make? (adapted from Ishida (2002)).